Introduction to QC part 1.

Slide 29, example of measuring $\langle QS \rangle_\psi$

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{2}$$

$$Q = Z_1 \quad , \quad S = \frac{-Z_2 - X_2}{\sqrt{2}}$$

$$\langle QS \rangle_\psi = \langle \psi | Q \otimes S | \psi \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \psi | -Z_1 Z_2 - Z_1 X_2 | \psi \rangle$$

$$= \frac{-1}{2\sqrt{2}} \Big[\big(\langle 01| - \langle 10|\big) Z_1 Z_2 \big(|01\rangle - |10\rangle\big)$$

$$+ \big(\langle 01| - \langle 10|\big) Z_1 X_2 \big(|01\rangle - |10\rangle\big) \Big]$$

$$= \frac{-1}{2\sqrt{2}} \Big[\big(\langle 01| - \langle 10|\big)\big(-|01\rangle + |10\rangle\big)$$

$$+ \big(\langle 01| - \langle 10|\big)\big(|00\rangle + |11\rangle\big) \Big]$$

$$= 1/\sqrt{2} .$$

The procedure is the same for the other expectation values

Slide 30: Quantum Teleportation

I sort the qubit from left to right (represented by the indices)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(\alpha|0\rangle_1\left(|00\rangle_{23}+|11\rangle_{23}\right) + \beta|1\rangle_1\left(|00\rangle_{23}+|11\rangle_{23}\right)\right)$$

$$|\psi_1\rangle = CNOT_{12}\,|\psi_0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle\left(|00\rangle+|11\rangle\right) + \beta|1\rangle\left(|10\rangle+|01\rangle\right)\right)$$

$$|\psi_2\rangle = H_1\,|\psi_1\rangle$$

$$= \frac{1}{2}\left(\alpha\left(|0\rangle+|1\rangle\right)\left(|00\rangle+|11\rangle\right) + \beta\left(|0\rangle-|1\rangle\right)\left(|10\rangle+|01\rangle\right)\right)$$

On va mesurer Qubit 1 & 2, rearrangeons l'équation :

$$|\psi_2\rangle = \frac{1}{2}\Big(|00\rangle\left(\alpha|0\rangle+\beta|1\rangle\right) + |01\rangle\left(\alpha|1\rangle+\beta|0\rangle\right) + |10\rangle\left(\alpha|0\rangle-\beta|1\rangle\right) + |11\rangle\left(\alpha|1\rangle-\beta|0\rangle\right)\Big)$$

After measurement, $|\psi_3\rangle$ has $\frac{1}{4}$ proba to end up in state

$M_1 M_2$		
00	$\longrightarrow \alpha\|0\rangle + \beta\|1\rangle$	$\rightarrow$ nothing
01	$\longrightarrow \alpha\|1\rangle + \beta\|0\rangle$	$\rightarrow X$
10	$\longrightarrow \alpha\|0\rangle - \beta\|1\rangle$	$\rightarrow Z$
11	$\longrightarrow \alpha\|1\rangle - \beta\|0\rangle$	$\rightarrow ZX$

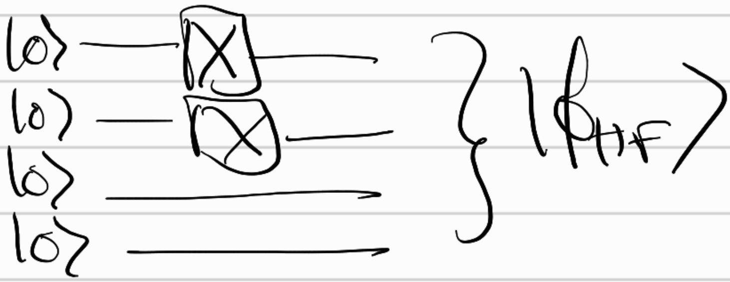
Introduction to QC part 2.

Slide 7 JW encoding

- HF state: $2e^-$ in 4 spin-orbs?

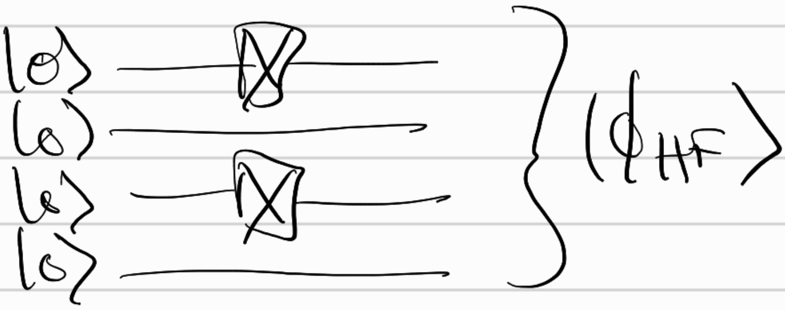
$|0011\rangle$

(Google).



$|0101\rangle$

(Qiskit)
IBM



It depends on conventions, so one code cannot be directly translated to another one.

Slide 8. QPE.

1) First register : $|0\rangle^{\otimes n} = |\underbrace{000\ldots0}_{n \text{ times}}\rangle$

Second register:

$$|\phi_{HF}\rangle = \sum_j a_j |\psi_j\rangle$$

We then apply Hadamard gate to each ancilla

2) First register : $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Second register: Still $|\phi\rangle = \sum_j a_j |\psi_j\rangle$

For now, they are separable states

We can write the full state as

$$\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |k\rangle \otimes \sum_j a_j |\psi_j\rangle$$

3) We apply Controlled-Unitary Gates between the two registers to create <u>entanglement</u>.

$$\sum_j a_j \left(\frac{1}{\sqrt{2}}\left(|0\rangle + e^{iE_j(2^{n-1})t}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{iE_j(2^{n-2})t}|1\rangle\right) \otimes \dots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{iE_j t}|1\rangle\right)\right) |\psi_j\rangle$$

This is called a phase-kickback

It can be rewritten as:

$$\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \sum_j a_j e^{iE_j k t} |k\rangle |\psi_j\rangle$$

(imagine $|1100\dots 0\rangle$, we have

$$e^{iE_j(2^{n-1}+2^{n-2})t}|1100\dots 0\rangle$$

where $2^{n-1}+2^{n-2}$ is exactly the integer associated to the binary.

Now, let's assume (see Zachary) that we know how to perform the Quantum analog of the Fourier Transform, i.e.

$$|j\rangle \longrightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^n} |k\rangle$$

It is Unitary so we can apply the inverse QFT to obtain

$$\sum_j a_j \, |\tilde{E_j t}\rangle |\psi_j\rangle$$

Finally measurement of the ancilla register gives the (classical) binary fraction corresponding to $\tilde{E_j t}$, which is an approximation of $E_j t$ with n bits of accuracy, with a probability of $|a_j|^2 = |\langle\phi|\psi_j\rangle|^2$.

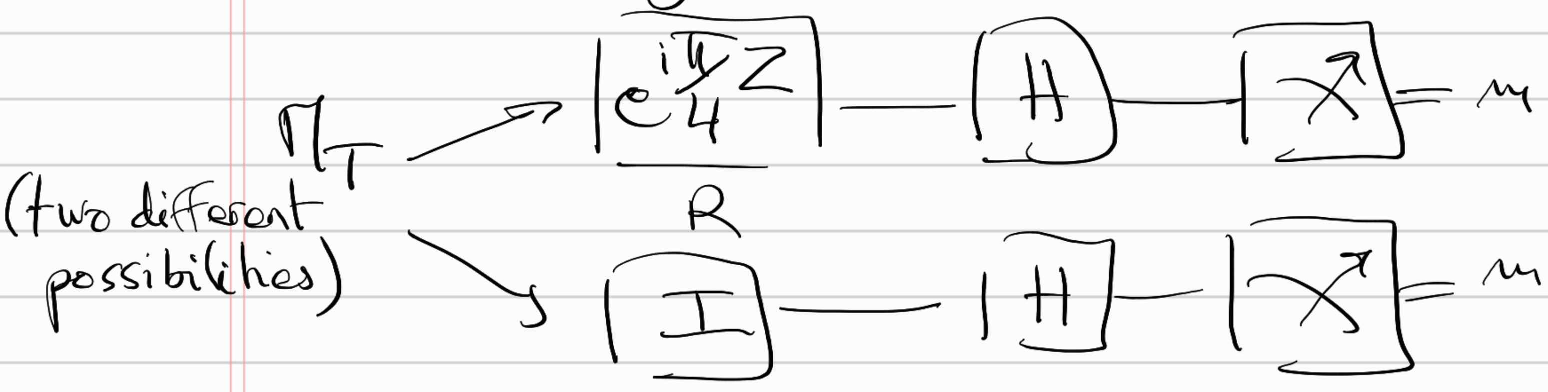
The system register collapses to the eigenstate $|\psi_j\rangle$ due to entanglement with the ancilla register

Slide 9 Single-ancilla QPE

As QPE, the state before applying the Tomography measurement gate M_T is

$$\sum_j a_j |\psi_j\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{iE_j kt}|1\rangle\right)$$

Let's see what we obtain after applying the M_T gate that is



So the two different M_T differ from the gate R which is either identity, or rotation along Z as $R = e^{i\frac{\pi}{4}Z} = \frac{1}{\sqrt{2}}(\mathbb{1} + iZ)$

Let us start with $R = \mathbb{1}$:

$$|S_{\mathbb{1}}\rangle = \frac{1}{2}\sum_j a_j |\psi_j\rangle\left(\left(1 + e^{iE_j kt}\right)|0\rangle + \left(1 - e^{iE_j kt}\right)|1\rangle\right),$$

Now, with $R = R_Z$. First apply R_Z:

$$\frac{1}{2}\sum_j a_j |\psi_j\rangle\left((1+i)|0\rangle + (1-i)e^{iE_j kt}|1\rangle\right)$$

And then the Hadamard gate:

$$|S_2\rangle = \frac{1}{2\sqrt{2}}\sum_j a_j|\psi_j\rangle\Big(\left(1+i+(1-i)e^{iE_jkt}\right)|0\rangle + \left(1+i-(1-i)e^{iE_jkt}\right)|1\rangle\Big)$$

Compute the probability:

$$P(m=0,\mathbb{1}) = \langle S_I|0\rangle\langle 0|S_I\rangle$$

$$= \frac{1}{4}\sum_{jl} a_j^* a_l \langle\psi_j|\psi_l\rangle\left(1+e^{-iE_jlt}\right)\left(1+e^{iE_lkt}\right)$$

And $\langle\psi_j|\psi_l\rangle = \delta_{jl}$

We have

$$P(m=0,\mathbb{1}) = \frac{1}{4}\sum_j |a_j|^2 \underbrace{\left(1+e^{-iE_jkt}\right)\left(1+e^{iE_jkt}\right)}_{2(1+\cos(E_jkt))}$$

$$= \frac{1}{2}\sum_j |a_j|^2\left(1+\cos(E_jkt)\right).$$

The other probabilities similarly read:

$$P(m=1,\mathbb{1}) = \frac{1}{2}\sum_j |a_j|^2\left(1-\cos(E_jkt)\right)$$

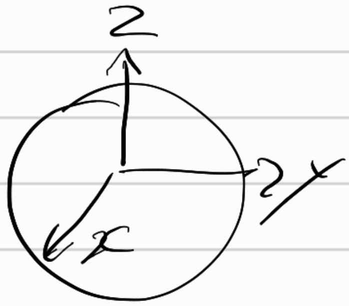
$$P(m=0,R_2) = \frac{1}{2}\sum_j |a_j|^2\left(1-\sin(E_jkt)\right)$$

$$P(m=1,R_2) = \frac{1}{2}\sum_j |a_j|^2\left(1+\sin(E_jkt)\right)$$

If we combine

$$g(k) = P(m=0, \mathbb{1}) - P(m=1, \mathbb{1})$$
$$+\, iP(m=1, R_2) - iP(m=0, R_2)$$
$$= \sum_j |a_j|^2 \left(\cos(E_j k t) + i \sin(E_j k t)\right)$$
$$= \sum_j |a_j|^2 e^{iE_j k t}$$

Slide 15. VQE Measurement



$\langle X \rangle_\psi = \langle \psi | X | \psi \rangle$ but we want to measure in the computational basis (observable Z)
Let's show that one has to rotate by a Hadamard gate:

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$HXH = \frac{1}{2}\Big[|0\rangle\left(\langle 0| + \langle 1|\right) + |1\rangle\left(\langle 0| - \langle 1|\right)\Big]$$
$$\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$
$$\Big[\left(|0\rangle + |1\rangle\right)\langle 0| + \left(|0\rangle - |1\rangle\right)\langle 1|\Big]$$

$$HXH = \frac{1}{2}\Big[|0\rangle(\langle 0| + \langle 1|) + |1\rangle(\langle 0| - \langle 1|)\Big]$$

$$(|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$= \frac{1}{2}\Big(|0\rangle(\langle 0| - \cancel{\langle 1|} + \langle 0| + \cancel{\langle 1|}) + |1\rangle(\cancel{\langle 0|} - \langle 1| - \cancel{\langle 0|} - \langle 1|)\Big)$$

$$= |0\rangle\langle 0| - |1\rangle\langle 1| = \hat{Z}$$

Hence, $\langle X \rangle_\psi = \langle Z \rangle_{\tilde{\psi}}$

where $|\tilde{\psi}\rangle = H|\psi\rangle$.

if $\langle Y \rangle$, we can show that

$$U = HSZ$$

if $\langle H \rangle$, $U = R_y\left(\frac{-\pi}{4}\right)$.