Self Consistent Random Phase Approximation in a three-level Lipkin model and the Goldstone mode

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Outline

I. Three-level Lipkin model

II. Self Consistent RPA (SCRPA)

III. Goldstone mode

IV. Conclusions

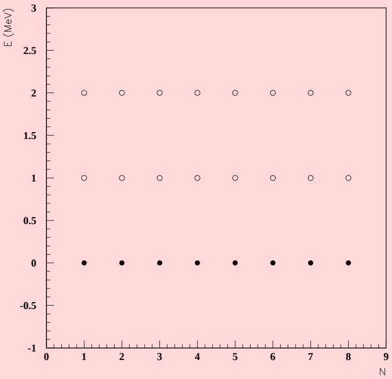
I. Three-level Lipkin model
SU(3) algebra

Three single particle levels α=**0,1,2**.

Level degeneracy on projection μ is $N = 2\Omega$ (number of particles).

0 is a hole level, filled with N particles in the ground state
1,2 are particle levels.

Figure:



THREE-LEVEL LIPKIN MODEL
with N=8 particles

Hamiltonian

$$H = \sum_{\alpha=0}^{2} \epsilon_\alpha K_{\alpha\alpha} - \frac{V}{2} \sum_{\alpha=1}^{2} (K_{\alpha 0} K_{\alpha 0} + K_{0\alpha} K_{0\alpha}) \ , \tag{1}$$

where **"quadrupole-like" operators** are defined as follows

$$K_{\alpha\beta} \equiv \sum_{\mu=1}^{N} c^\dagger_{\alpha\mu} c_{\beta\mu} \ . \tag{2}$$

$c^\dagger_{\alpha\mu}$ is a fermion creation operator on α-th level.

Commutation rules

$$[K_{\alpha\beta}, K_{\gamma\delta}] = \delta_{\beta\gamma} K_{\alpha\delta} - \delta_{\alpha\delta} K_{\gamma\beta} \ . \tag{3}$$

Continuously broken symmetry

appears when $\epsilon_1 = \epsilon_2$.

The angular momentum projection operator

$$\hat{L}_0 = i(K_{21} - K_{12}) \,, \tag{4}$$

commutes with the Hamiltonian, i.e.

$$[H, \hat{L}_0] = 0 \,. \tag{5}$$

Will will show that SCRPA exhibits a **Goldstone mode**
with a vanishing energy, as this is also the case with standard RPA.
That this property is conserved has already been announced
by D. Rowe in Rev. Mod. Phys. **40**, 153 (1968),
but never has been explicitly verified.

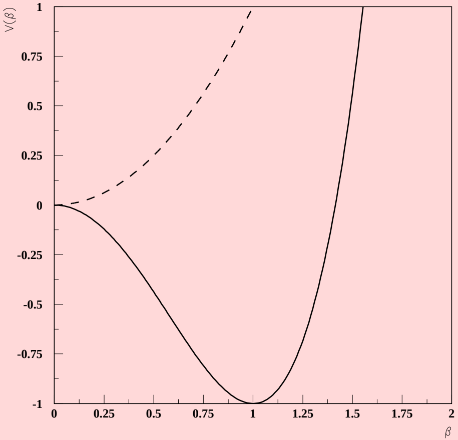


Figure:

EXAMPLE OF THE SYMMETRY BREAKING:
spherical $\rightarrow$ deformed ground state in intrinsic β-coordinate for any angle (”Mexican hat”)

II. Self consistent RPA (SCRPA)

A. Single particle mean field (MF) basis

B. SCRPA equations

C. Generalized MF equations

A. Single particle mean field (MF) basis

is defined by the following rotation of the basis operators

$$a^{\dagger}_{k\mu} = \sum_{\alpha=0}^{n} C_{k\alpha} c^{\dagger}_{\alpha\mu} \ , \tag{6}$$

Spherical basis is defined by a diagonal transformation: $C_{k\alpha} = \delta_{k\alpha}$
Deformed basis is given by: $C_{k\alpha} \neq \delta_{k\alpha}$

”Quadrupole-like” operators

become in this basis

$$K_{\alpha\beta} \rightarrow A_{ij} \ = \ \sum_{\mu=1}^{N} a^{\dagger}_{i\mu} a_{j\mu} \ . \tag{7}$$

SCRPA creation operators

$$Q_\nu^\dagger = \sum_{i>j=0}^{2} (X_{ij}^\nu \delta Q_{ij}^\dagger - Y_{ij}^\nu \delta Q_{ij}) \,, \qquad (8)$$

are written in terms of the normalized generators

$$\begin{aligned} \delta Q_{ij}^\dagger &= N_{ij}^{-1/2} A_{ij} \\ \delta Q_{ij} &= \delta Q_{ji}^\dagger = N_{ij}^{-1/2} A_{ji} \,. \end{aligned} \qquad (9)$$

1,0 and **2,0** are particle-hole (ph) terms, while
2,1 is the particle-particle (pp) scattering term
which do not appear within the standard RPA.

Normalisation factor

is given by the mean value of the following commutator on the correlated vacuum

$$\begin{aligned} \langle 0|[A_{ji}, A_{i'j'}]|0\rangle &= \delta_{ii'}\delta_{jj'}(\langle 0|A_{ii}|0\rangle - \langle 0|A_{jj}|0\rangle) \\ &\equiv \delta_{ii'}\delta_{jj'}N_{ij} \ . \end{aligned} \quad (10)$$

The pp scattering term (21**) has small values and in standard RPA is neglected**.

Excited states

are defined by the SCRPA creation operators

$$|\nu\rangle = Q_\nu^\dagger|0\rangle \ . \tag{11}$$

In the derivation of SCRPA equations one supposes the existence of a correlated ground state such that

$$Q_\nu|0\rangle = 0 \ . \tag{12}$$

This equation has solution only for simple models depending upon only one parameter Z=Y/X (two level Lipkin model).

B. SCRPA equations

$$\langle 0|[\delta Q_\nu, [H, Q_\nu^\dagger]]|0\rangle = \omega_\nu \langle 0|[\delta Q_\nu, Q_\nu^\dagger]|0\rangle \ . \tag{13}$$

have the following matrix form

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & -\mathcal{A}^* \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} = \omega_\nu \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} \ . \tag{14}$$

The SCRPA matrices

are given by the well-known relations

$$\begin{aligned} \mathcal{A}_{ij,i'j'} &= \langle 0| \left[\delta Q_{ij}, \left[H, \delta Q_{i'j'}^\dagger\right]\right] |0\rangle \\ \mathcal{B}_{ij,i'j'} &= -\langle 0| \left[\delta Q_{ij}, \left[H, \delta Q_{i'j'}\right]\right] |0\rangle = -\mathcal{A}_{ij,j'i'} \ . \end{aligned} \tag{15}$$

C. Generalized MF equations

$$\langle 0|[H, Q_\nu]|0\rangle = \langle 0|[H, Q_\nu^\dagger]|0\rangle = 0 \; . \tag{16}$$

are of the form

$$\sum_m \mathcal{H}_{nm} C_m = E_\alpha \langle 0|A_{\alpha\alpha}|0\rangle C_{n\alpha} \; , \tag{17}$$

SCRPA and MF matrix elements contain

one-body and two-body densities

$$\langle 0|A_{ii}|0\rangle \,, \qquad \langle 0|A_{ij}A_{kl}|0\rangle \,. \tag{18}$$

The densities are computed in terms of SCRPA amplitudes X and Y by inverting SCRPA operators

$$\delta Q^{\dagger}_{ij} = N_{ij}^{-1/2} A^{\dagger}_{ij} = \sum_{\nu} \left(X^{\nu}_{ij} Q^{\dagger}_{\nu} + X^{\nu}_{ij} Q_{\nu} \right) \,, \tag{19}$$

and by using the relations

$$Q_{\nu}|0\rangle = \langle 0|Q^{\dagger}_{\nu} = 0 \,. \tag{20}$$

The SCRPA and generalized MF equations are nonlinear. They are solved iteratively.

Technical details

One obtains for two body densities

$$
\begin{aligned}
\langle 0|A_{ij}A_{kl}|0\rangle &= N_{ij}^{1/2}N_{kl}^{1/2}\sum_{\nu} Y_{ij}^{\nu}X_{kl}^{\nu}\ ,\ i>j,\ k>l \\
\langle 0|A_{ji}A_{lk}|0\rangle &= N_{ij}^{1/2}N_{kl}^{1/2}\sum_{\nu} X_{ij}^{\nu}Y_{kl}^{\nu} \\
\langle 0|A_{ij}A_{lk}|0\rangle &= N_{ij}^{1/2}N_{kl}^{1/2}\sum_{\nu} Y_{ij}^{\nu}Y_{kl}^{\nu} \\
\langle 0|A_{ji}A_{kl}|0\rangle &= N_{ij}^{1/2}N_{kl}^{1/2}\sum_{\nu} X_{ij}^{\nu}X_{kl}^{\nu}\ ,
\end{aligned} \tag{21}
$$

where normalisation factors are given as differences of one body densities.

In order to compute one body densities we consider the expansion

$$A_{mm} = \sum_{n_1 n_2 = 0}^{N} c_{n_1 n_2}(m) A_{10}^{n_1} A_{20}^{n_2} A_{02}^{n_2} A_{01}^{n_1} \,, \tag{22}$$

To find $c_{n_1 n_2}(m)$ we compute the expectation values on the correlated vacuum $|0\rangle$ and use the inversion of the RPA operator.
One finally obtains a nonlinear system of equations, determining the normalisation factors N_{10}, N_{20}.

One obtains for the leading terms of one body densities with $n_1 + n_2 \leq 1$ the following simple analytical expressions

$$\begin{aligned} \langle 0|A_{mm}|0\rangle &= \left[y_{mm} + \frac{y_{11}y_{22}}{N}\right] \\ &\times \left[1 + \frac{2}{N}(y_{11} + y_{22}) + \frac{3}{N^2}y_{11}y_{22}\right]^{-1} \\ \langle 0|A_{00}|0\rangle &= N - \langle 0|A_{11}|0\rangle - \langle 0|A_{22}|0\rangle \ , \end{aligned} \tag{23}$$

where

$$y_{mn} = \sum_{\nu} Y^{\nu}_{m0} Y^{\nu}_{n0} \ . \tag{24}$$

Mean Field transformation

can be written as a product of two rotations

$$C_{k\alpha} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & \sin\phi\cos\psi & \sin\phi\sin\psi \\ -\sin\phi & \cos\phi\cos\psi & \cos\phi\sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix} . \qquad (25)$$

Hamiltonian mean value

on the non-correlated mean field HF vacuum is given by

$$\begin{aligned} \langle HF|H|HF\rangle &= N\epsilon[e_0\cos^2\phi + e_1\sin^2\phi\cos^2\psi \\ &+ e_2\sin^2\phi\sin^2\psi - \chi\sin^2\phi\cos^2\phi] \ , \end{aligned} \tag{26}$$

where we introduced the following dimensionless notations

$$e_k = \frac{\epsilon_k}{\epsilon}, \quad \chi = \frac{V(N-1)}{\epsilon} \ . \tag{27}$$

Hamiltonian minima

Spherical minimum

$$1) \qquad \phi = 0, \quad \psi = 0\ , \quad \chi < e_1 - e_0\ , \tag{28}$$

Deformed minimum

$$2) \qquad \cos 2\phi = \frac{e_1 - e_0}{\chi}, \quad \psi = 0, \quad \chi > e_1 - e_0\ . \tag{29}$$

Moreover, our calculations have shown that for any MF minimum one obtains $\psi = 0$, independent of which kind of vacuum (correlated or not) we use to estimate the expectation values.

Standard RPA

Vacuum state is the Hartree-Fock ground state

$$|0\rangle = |HF\rangle \ . \tag{30}$$

We fix the origin of the particle spectrum with $e_0 = 0$.
If $\Delta\epsilon \equiv \epsilon_2 - \epsilon_1 = 0$, for the values of the strength $\chi > e_1$, in the **deformed region**, i.e. with $\phi \neq 0$ given by HF minimum, one obtains a **Goldstone mode**.

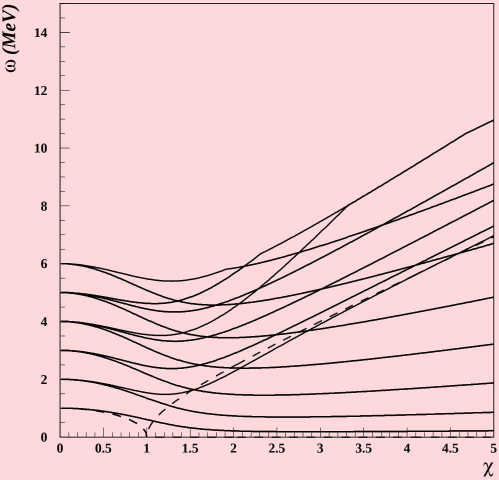


Figure:

EXACT SOLUTION

versus the strength parameter χ for $N = 20, \epsilon_1 = 0, \epsilon_2 = \epsilon_3 = 1 MeV$. By dashes are given standard RPA values. Notice the appearance of a zero mode solution (Goldstone mode) beyond χ=1.

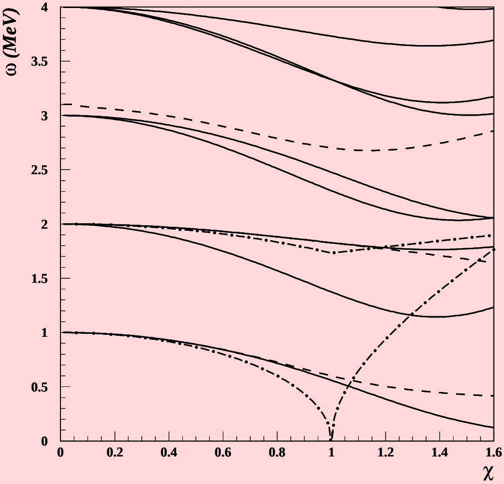


Figure:

SCRPA IN THE SPHERICAL REGION

versus the strength parameter χ for $N = 20$ and $e_0 = 0,\ e_1 = 1,\ e_2 = 2$ (dashed lines). By solid lines are given the lowest exact eigenvalues and by dot-dashes the standard RPA energies.

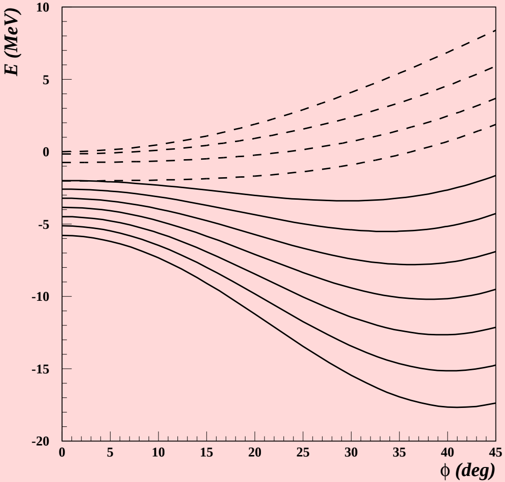


Figure:

TRANSITION FROM SPHERICAL TO THE DEFORMED REGION

The SCRPA expectation value of the Hamiltonian versus the angle ϕ, for $N = 20$ and different values of the strength parameter χ (from the top of the figure, $\chi = 0, 0.5, ..., 5$).

III. Goldstone mode

The commutation relation

$$[H, L_0] = 0 \ , \tag{31}$$

can be seen as an

RPA equation with zero energy ω=0

$$[H, L_0] = \omega L_0 \ . \tag{32}$$

Thus, SCRPA will exhibit a Goldstone mode, as this is also the case with standard RPA.
That this property is conserved has already been announced by Rowe, but never has been explicitly verified.

As a matter of fact we checked that **for an SCRPA operator restricted to *ph* and *hp* configurations the Goldstone mode does NOT come at zero energy**. The reason for this is simple: usually a symmetry operator contains also (*hh*) and (*pp*) configurations, and without them, it is atrophiated and SCRPA fails to produce a zero mode.

In standard RPA this does not matter because *hh* and *pp* configurations decouple. Beyond standard RPA it matters and **the inclusion of scattering terms produces the Goldstone mode**.

This is the reason why we think that the three-level Lipkin Hamiltonian is adequate since it can be studied in the limit $\delta\epsilon = \epsilon_2 - \epsilon_1 \rightarrow 0$ where the spontaneously broken symmetry shows up.

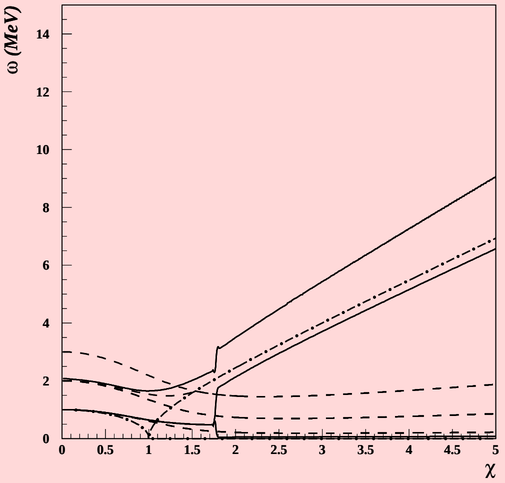


Figure:

GOLDSTONE MODE

SCRPA excitation energies versus the strength parameter χ, for $N = 20$, $\Delta\epsilon = 0.001$ *MeV* (full line). By dashes are given the lowest exact eigenvalues and by dot-dashes the standard RPA energies.

IV. Conclusions

1. The three-level Lipkin model has the advantage of allowing for a **continuously broken symmetry** on the mean field level with the appearance of a **Goldstone mode**.

2. The **RPA operator** should contain, in addition to the usual ph components $a_k^\dagger a_0$, also the so-called anomalous or **scattering terms** $a_2^\dagger a_1$.

3. Therefore the present formulation of SCRPA allows to mentain all the **formal and desirable properties of standard RPA:** **conservation laws, sum rules are fulfilled**

Acknowledgments

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